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Functional formalism for ordered electromagnetic field operators

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Abstract. The functional formalism, providing a unified treatment of the optical coherence theory, is developed for arbitrarily ordered field operators. Multimode ordered expansions of operators in terms of boson amplitude operators as well as integral expansions are discussed. They form a basis of associating a class of s-ordered probability functionals with the density operator of the field. As an application, the distributions of photons and photoelectrons are studied. The generalized photo-detection equation of Perina and Horak is rederived by using the functional approach. A formula expressing the photocount generating function through the characteristic functional of the field is derived.

1. Introduction

In the quantum theory of coherence, one considers statistical averages of products of the field operators ordered according to a certain rule. For example, the photocount statistics is formulated in terms of normally ordered products if the photoelectric detector is used as a measuring device, but in the case of the quantum counter anti-normal order is involved. The evaluation of expectation values of ordered products may be simplified by assigning a correspondence between the field operators and c-number functions and also by associating a weight function with the density operator describing the state of the field. General ordering theorems were presented by Agarwal and Wolf (1968). Cahill and Glauber (1969a, b) analysed various ways of defining correspondences between operators and functions.

When the appropriate weight function specifying the density operator is known, a multifold probability distribution P_N of the field can be calculated. This permits one to find the Nth order correlation functions. In the limit, $N \to \infty$, P_N becomes a probability functional characterizing the field completely.

In mathematical terms, the probability functional defines a measure on a probability measure space in which random processes become measurable functions. This approach, however attractive, involves difficulties related to the fact that there is no direct analogue in function space to the Euclidean volume in N dimensions. The way out of this predicament consists in constructing isotropic, centred Gaussian integrals on Hilbert space (Segal 1965). The method of functional integration are still not developed to a point where techniques for solving explicit problems are available. The only integrals that lend themselves to calculation are Gaussian integrals or those that can be reduced to them by functional transformations of variables. However, in many problems, particularly in quantum field theory and statistical mechanics, the functional approach has been very useful. For our purpose an intuitive understanding of the functional integration, such as given in the work of Symanzik (1954) will be sufficient. In § 4, we deal explicitly with Gaussian integrals. More complicated cases can be interpreted as limits of multifold integrals where the number of variables tends to infinity. Throughout our exposition, we assume, following Symanzik, the existence of the functional Fourier transforms.

For the normally ordered field operators a relation between the weight function of the density operator and the characteristic functional of the field (functional Fourier transform of the probability functional) was derived previously (Zardecki 1970). In this paper, we extend the functional formalism for an arbitrary kind of ordering. To this end, we analyse, first of all, ordered operator expansions in the multi-mode case. As an application of the functional approach, we shall rederive the generalized photodetection equation of Perina and Horak. The main new result, playing a role in physical applications, will be a general formula expressing the generating function of the photoelectron distribution in terms of the ordered characteristic functional of the field.

2. Multi-mode ordered operator expansions

The multi-mode displacement operator is defined as the product of one-mode displacement operators

$$D(\{\alpha_k\}) = \prod_k D(\alpha_k) = \prod_k \exp(\alpha_k a_k^\dagger - \alpha_k^* a_k), \tag{2.1}$$

where a_k and $a_k^\dagger$ are the boson annihilation and creation operators of the kth mode, obeying the standard commutation relations

$$[a_k, a_{k'}^\dagger] = \delta_{kk'}. \tag{2.2}$$

The displacement operators are orthogonal in the following sense

$$\mathrm{Tr}[D(\{\alpha_k\})D^{-1}(\{\beta_k\})] = \prod_k \pi\delta^{(2)}(\alpha_k - \beta_k). \tag{2.3}$$

In addition, an arbitrary operator F whose Hilbert–Schmidt norm is finite, ie $[\mathrm{Tr}(F^\dagger F)]^{1/2} < \infty$, can be expanded in terms of the displacement operators

$$F = \int f(\{\xi_k\})D^{-1}(\{\xi_k\})\,\mathrm{d}^2\{\xi_k/\pi\} \tag{2.4}$$

the weight function $f(\{\xi_k\})$ being given by

$$f(\{\xi_k\}) = \mathrm{Tr}[FD(\{\xi_k\})]. \tag{2.5}$$

In order to prove the relations (2.3)–(2.5) and other theorems of this section, one can proceed in complete analogy to the single-mode case (Cahill and Glauber 1969a, b). In the case of finite number of modes, $k = 1, 2, \ldots, M$, an integral such as (2.4) is well defined. Some complications arising for an infinite number of degrees of freedom, which is usually the case where the electromagnetic field is involved, can be circumvented by applying a limiting procedure (Klauder and Sudarshan 1968). To this end, the condition

$$\sum_{k=1}^{\infty} |\alpha_k|^2 < \infty$$

for the allowed sequences $\{\alpha_k\}$ is adopted. The integrals over infinitely many variables are to be understood as the limits $M \to \infty$, over truncated sequences $\{\alpha_{k,M}\}$, where $\alpha_{k,M} \equiv \alpha_k$, for $k \leqslant M$, and $\alpha_{k,M} = 0$, for $k > M$.

We define the s-ordered displacement operator as

$$D(\{\alpha_k\}, s) = \prod_k D(\alpha_k) \exp(\tfrac{1}{2}s|\alpha_k|^2), \tag{2.6}$$

from which the s-ordered products of the creation and annihilation operators can be generated. If we expand each factor in (2.6), we can write

$$D(\{\alpha_k\}, s) = \prod_k \sum_{n_k, m_k = 0}^{\infty} \frac{\alpha_k^{n_k}(-\alpha_k^*)^{m_k}}{n_k! m_k!} \{(a_k^\dagger)^{n_k} a_k^{m_k}\}_s. \tag{2.7}$$

Since the creation and annihilation operators belonging to different modes commute, this can be also written as

$$D(\{\alpha_k\}, s) = \sum_{\{n_k\}\{m_k\}} \prod_k \frac{\alpha_k^{n_k}(-\alpha_k^*)^{m_k}}{n_k! m_k!} \left\{\prod_k (a_k^\dagger)^{n_k} a_k^{m_k}\right\}_s. \tag{2.8}$$

The multimode s-ordered product in (2.8) is defined equivalently as the multiple derivative of (2.6)

$$\begin{aligned} \{[a^\dagger]^n [a]^m\}_s &\equiv \left\{\prod_k (a_k^\dagger)^{n_k} a_k^{m_k}\right\}_s \\ &= \prod_k \frac{\partial^{n_k + m_k}}{\partial \alpha_k^{n_k} \partial(-\alpha_k^*)^{m_k}} D(\{\alpha_k\}, s)|_{\{\alpha_k\} = 0} \end{aligned} \tag{2.9}$$

evaluated at $\{\alpha_k\} = 0$.

Let us now consider the problem of expanding an operator F as a power series in the s-ordered products

$$F = \sum_{\{n_k\}\{m_k\}} f(\{n_k\}, \{m_k\}, s)\{[a^\dagger]^n [a]^m\}_s. \tag{2.10}$$

Broadly speaking, the expansion is convergent for orders that are closer to normal order ($s = 1$) than to symmetric order ($s = 0$). The expansion coefficients are given explicitly as

$$f(\{n_k\}, \{m_k\}, s) = \prod_k (n_k! m_k!)^{-1} \int \operatorname{Tr}[F D(\{\xi_k\}, -s)](-\xi_k)^{n_k}(\xi_k^*)^{m_k} \, \mathrm{d}^2\{\xi_k/\pi\}. \tag{2.11}$$

Integral expansions for operators are introduced by means of the operator $T(\{\alpha_k\}, s)$ which is a complex Fourier transform of the operator $D(\{\alpha_k\}, s)$

$$T(\{\alpha_k\}, s) = \int \exp\left(\sum_k (\alpha_k \xi_k^* - \alpha_k^* \xi_k\right) D(\{\xi_k\}, s) \, \mathrm{d}^2\{\xi_k/\pi\}. \tag{2.12}$$

The integral expansion for an arbitrary operator is

$$F = \int f(\{\alpha_k\}, s) T(\{\alpha_k\}, s) \, \mathrm{d}^2\{\alpha_k/\pi\} \tag{2.13}$$

where the weight function f has the form

$$f(\{\alpha_k\}, -s) = \operatorname{Tr}[F T(\{\alpha_k\}, -s)]. \tag{2.14}$$

The usefulness of these expressions comes from the relationship between the coefficients (2.11) and the weight function (2.14). The latter is the same function of the numbers $\{\alpha_k^*\}$ and $\{\alpha_k\}$ as an operator F, represented by $f(\{\alpha_k\}, -s)$, is of the operators $\{a_k^\dagger\}$ and $\{a_k\}$. In other words, we have

$$f(\{\alpha_k\}, -s) = \sum_{\{n_k\},\{m_k\}} f(\{n_k\}, \{m_k\}, s) \prod_k (\alpha_k^*)^{n_k} \alpha_k^{m_k} \tag{2.15}$$

which means that the operator–function correspondence

$$F \underset{s}{\leftrightarrow} f(\{\alpha_k\}, -s) \tag{2.16}$$

involves the interchange

$$\left\{\prod_k (a_k^\dagger)^{n_k} a_k^{m_k}\right\}_s \underset{s}{\leftrightarrow} \prod_k (\alpha_k^*)^{n_k} \alpha_k^{m_k}. \tag{2.17}$$

This correspondence, referred to as the $C(s)$ correspondence, will form the basis for applications. We note, in particular,

$$D(\{\xi_k\}, s) \underset{s}{\leftrightarrow} \prod_k \exp(\xi_k \alpha_k^* - \xi_k^* \alpha_k) \tag{2.18}$$

which will permit later a simple representation of the characteristic functional.

3. Characteristic functional and probability functional

The theory will be formulated in terms of the positive and negative frequency field operators $A_\mu^{(+)}(x)$ and $A_\mu^{(-)}(x)$. These are given as the linear combinations of the annihilation and creation operators, respectively,

$$\begin{aligned} A_\mu^{(+)}(x) &= \sum_k v_\mu(x, k) a_k, \\ A_\mu^{(-)}(x) &= \sum_k v_\mu^*(x, k) a_k^\dagger \end{aligned} \tag{3.1}$$

where the functions $v_\mu(x, k)$ depend on a specific choice of the operators $A_\mu(x)$. For example, if $A_\mu(x)$ represents the vector potential obeying the transversality condition, we have (in Heaviside units)

$$v_\mu(x, k) = c\left(\frac{\hbar}{2\omega_k}\right)^{1/2} U_{k\mu}(\boldsymbol{r})\, \mathrm{e}^{-\mathrm{i}\omega_k t} \tag{3.2}$$

where $\omega_k = ck$, and the mode functions $U_k(\boldsymbol{r})$ form an orthonormal and complete set of solutions to the Helmholtz equation.

If $A_\mu(x)$ denotes the detection operator (Mandel 1964, 1966), then

$$v_\mu(x, k) = L^{-3/2}\, e_\mu^{(s)}(\boldsymbol{k})\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}.\boldsymbol{r}} \tag{3.3}$$

where the superscript s denotes the polarization index, and $k = (\boldsymbol{k}, s)$.

From the commutation relations (2.2), it follows that the commutator

$$\begin{aligned} K_{\mu\nu}(x, x') &\equiv [A_\mu^{(+)}(x), A_\nu^{(-)}(x')] \\ &= \sum_k v_\mu(x, k) v_\nu^*(x'k) \end{aligned} \tag{3.4}$$

is a c number.

The state of radiation being characterized by the density operator ρ, we define the s-ordered quantum characteristic functional of the electromagnetic field as follows.

$$\Phi[\zeta(x)|s] = \mathrm{Tr}(\rho \exp\{(\zeta, A^{(-)}) - (\zeta^*, A^{(+)}) + \tfrac{1}{2}sK[\zeta^*, \zeta]\}) \tag{3.5}$$

where the abbreviations

$$\begin{aligned}(\zeta, A^{(-)}) &= \int \zeta_\mu(x) A_\mu^{(-)}(x)\,\mathrm{d}^4x \\ (\zeta^*, A^{(+)}) &= \int \zeta_\mu^*(x) A_\mu^{(+)}(x)\,\mathrm{d}^4x \\ K[\zeta^*, \zeta] &= \iint \zeta_\mu^*(x) K_{\mu\nu}(x, x')\zeta_\nu(x')\,\mathrm{d}^4x\,\mathrm{d}^4x'\end{aligned} \tag{3.6}$$

have been used. In equations (3.6) summation over repeated indices is understood. In view of equations (3.1) and (3.4), we can regard the characteristic functional as a function of a denumerable set of variables $\{\zeta_k\}$

$$\begin{aligned}\Phi(\{\zeta_k\}, s) &= \mathrm{Tr}\left(\rho \prod_k \exp(\zeta_k a_k^\dagger - \zeta_k^* a_k + \tfrac{1}{2}s|\zeta_k|^2)\right) \\ &= \mathrm{Tr}[\rho D(\{\zeta_k\}, s)]\end{aligned} \tag{3.7}$$

where

$$\zeta_k = \int v_\mu^*(x, k)\zeta_\mu(x)\,\mathrm{d}^4x. \tag{3.8}$$

Thus, the characteristic functional can also be defined as an expectation value of the displacement operator.

Let the density operators possess an integral representation of the form (2.13)

$$\rho = \int W(\{\alpha_k\}, s)T(\{\alpha_k\}, -s)\,\mathrm{d}^2\{\alpha_k/\pi\}. \tag{3.9}$$

The weight function $W(\{\alpha_k, s)$ associated with ρ by $C(-s)$ correspondence is given as the trace

$$W(\{\alpha_k\}, s) = \mathrm{Tr}[\rho T(\{\alpha_k\}, s)]. \tag{3.10}$$

Inserting (3.9) into (3.5), we obtain

$$\Phi[\zeta(x)|s] = \int W(\{\alpha_k\}, s)\,\mathrm{Tr}[T(\{\alpha_k\}, -s)\exp((\zeta, A^{(-)}) - (\zeta^*, A^{(+)}) + \tfrac{1}{2}sK[\zeta^*, \zeta])]\,\mathrm{d}^2\{\alpha_k/\pi\}. \tag{3.11}$$

It follows from equation (2.18), and from the linearity of the field operators in creation and annihilation operators, that the weight function associated by the $C(s)$ correspondence with the exponential operator in the integrand of (3.11) is found by the interchange

$$\exp\{(\zeta, A^{(-)}) - (\zeta^*, A^{(+)}) + \tfrac{1}{2}sK[\zeta^*, \zeta]\} \underset{s}{\leftrightarrow} \exp[(\zeta, V^*) - (\zeta^*, V)]. \tag{3.12}$$

The functions $V_\mu(x)$ and $V_\mu^*(x)$ arise from the operators $A_\mu^{(+)}(x)$ and $A^{(-)}(x)$ by replacing the operators a_k and $a_k^\dagger$ in (3.1) by their eigenvalues α_k and α_k^*, respectively

$$\begin{aligned} V_\mu(x) &\equiv V_\mu(x, \{\alpha_k\}) = \sum_k v_\mu(x, k)\alpha_k \\ V_\mu(x) &\equiv V_\mu^*(x, \{\alpha_k\}) = \sum_k v_\mu^*(x, k)\alpha_k^*. \end{aligned} \tag{3.13}$$

By virtue of (2.14), (3.11), and (3.12), we obtain

$$\Phi[\zeta(x)|s] = \int W(\{\alpha_k\}, s) \exp\left(\int \zeta_\mu(x)V_\mu^*(x, \{\alpha_k\} \, d^4x - \int \zeta_\mu^*(x)V_\mu(x, \{\alpha_k\}) \, d^4x\right) d^2\{\alpha_k/\pi\}. \tag{3.14}$$

Equivalently, equation (3.14) can be written in the form

$$\Phi(\{\zeta_k\}, s) = \int W(\{\alpha_k\}, s) \exp\left(\sum_k (\zeta_k\alpha_k^* - \zeta_k^*\alpha_k)\right) d^2\{\alpha_k/\pi\} \tag{3.15}$$

where ζ_k are defined by equation (3.8). For $s = 1$, $W(\{\alpha_k\}, s)$ is to be identified with the weight function $P\{\alpha_k\}$ specifying the diagonal representation of the density operator (P representation), and from equation (3.15) we get a relation between $P\{\alpha_k\}$ and the normally ordered characteristic functional. This relation was already derived earlier within the framework of functional formalism developed for normally ordered electromagnetic-field operators (Zardecki 1970).

By the Fourier inversion formula, we obtain from equation (3.15):

$$W(\{\alpha_k\}, s) = \int \Phi(\{\zeta_k\}, s) \exp\left(\sum_k (\alpha_k\zeta_k^* - \alpha_k^*\zeta_k)\right) d^2\{\alpha_k/\pi\} \tag{3.16}$$

which establishes an equivalence of statistical descriptions based on the weight function and on the characteristic functional. For an infinite number of modes the Fourier transformation is obtained from an ordinary M-dimensional Fourier integral by the limiting transition $M \to \infty$. A class of s-ordered probability functionals of the field can be introduced by making use of the delta functional

$$P[V(x)|s] = \int W(\{\alpha_k\}, s)\delta^{(2)}\left[V(x) - \sum_k v(x, k)\alpha_k\right] d^2\{\alpha_k/\pi\}. \tag{3.17}$$

As usual, $\delta^{(2)}[V(x)]$ denotes a complex functional Fourier transform of unity.

$$\delta^{(2)}[V(x)] = \int \exp((V, \zeta^*) - (V^*, \zeta)) \, d^2[\zeta/\pi] \tag{3.18}$$

where $d^2\zeta$ is an integration element in function space over the real and imaginary parts of the function $\zeta(x)$.

With the help of the probability functional (3.17) the characteristic functional (3.14) can be written as a functional integral

$$\Phi[\zeta|s] = \int P[V|s] \exp((\zeta, V^*) - (\zeta^*, V)) \, d^2V. \tag{3.19}$$

We observe that there are distinct characteristic functionals and probability functionals corresponding to each value of the order parameter s. The problem of existence of the probability functional is related to the possibility of construction of the weight function $W(\{\alpha_k\}, s)$.

By applying the Fourier inversion formula, we obtain from (3.19).

$$P[V|s] = \int \Phi[\zeta|s] \exp((V, \zeta^*) - (V^*, \zeta))\, \mathrm{d}^2[\zeta/\pi^2] \tag{3.20}$$

which is another way of introducing the probability functional. The pair of functional Fourier transforms (3.19) and (3.20) exists if one of the functionals Φ or P is square integrable (Symanzik 1954).

The quantum correlation function is defined as an expectation value of the s-ordered product of the field operators

$$G_s^{(N,M)}(x_1, \ldots, x_N; x_{N+1}, \ldots, x_{N+M}) = \mathrm{Tr}[\rho\{A^{(-)}(x_1) \ldots A^{(-)}(x_N) A^{(+)}(x_{N+1}) \ldots A^{(+)}(x_{N+M})\}_s]. \tag{3.21}$$

For the sake of brevity we use now x to denote both the space–time point r, t and the tensor index μ. Equation (3.21) can also be written as a functional integral

$$G_s^{(N,M)}(x_1, \ldots, x_N; x_{N+1}, \ldots, x_{N+M}) = \int P[V|s] V^*(x_1) \ldots V^*(x_N) V(x_{N+1}) \ldots V(x_{N+M})\, \mathrm{d}^2 V. \tag{3.22}$$

By functional differentiation of the characteristic functional, the entire set of s-ordered correlation functions is derived

$$G_s^{(N,M)}(x_1, \ldots, x_N; x_{N+1}, \ldots, x_{N+M}) = \prod_{j=1}^{N} \frac{\delta}{\delta\zeta(x_j)} \prod_{j=n+1}^{N+M} \frac{\delta}{\delta(-\zeta^*(x_j))} \Phi[\zeta|s]|_{\zeta=0}. \tag{3.23}$$

The relation between an arbitrary s-ordered correlation function and the t-ordered correlation function is derived from the relation between the corresponding characteristic functionals. We write

$$\Phi[\zeta(x)|s] = \exp\{\tfrac{1}{2}(s-t)K[\zeta^*, \zeta]\}\Phi[\zeta(x)|t] \tag{3.24}$$

which yields on differentiation with respect to $\zeta(x_i)$, $i = 1, \ldots, N$, and using the Leibniz rule for the Nth derivative of a product

$$\frac{\delta^N \Phi[\zeta|s]}{\delta\zeta(x_1) \ldots \delta\zeta(x_N)} = \sum_{l=0}^{N} \binom{N}{l} \left(\frac{s-t}{2}\right)^l \left(\prod_{j=1}^{l} \int \zeta^*(x') K(x', x_j)\, \mathrm{d}x'\right) \exp\{\tfrac{1}{2}(s-t)K[\zeta^*, \zeta]\} \times \frac{\delta^{N-l}\Phi[\zeta|t]}{\delta\zeta(x_{l+1}) \ldots \delta\zeta(x_N)}. \tag{3.25}$$

This is to be differentiated now with respect to $(-\zeta^*(x_i))$, $i = N+1, \ldots, N+M$.

Since $\zeta(x)$ is ultimately put equal to zero, we obtain the formula

$$G_s^{(N,M)}(x_1, \ldots, x_N; x_{N+1}, \ldots, x_{N+M}) = \sum_{l=0}^{(N,M)_{\min}} \binom{N}{l} \binom{M}{l} \left(\frac{t-s}{2}\right)^l \sum_P \prod_{j=1}^{l} K(x_{P(j)}, x_j) \times G_t^{(N-l, M-l)}(x_{l-1}, \ldots, x_{N'}, x_{N+l+1}, \ldots, x_{N+M}) \tag{3.26}$$

where the symbol $(N, M)_{\min}$ denotes the smaller of the integers N and M, and the second summation runs over all permutations of the coordinates $x_1, \ldots, x_l$.

4. Examples: thermal and coherent fields

The weight functions specifying the density operator corresponding to different values of the order parameter are related by a convolution law which is a simple generalization of the result obtained by Cahill and Glauber (1969b) in the case of one-mode field. If $\operatorname{Re} s < \operatorname{Re} t$, we have

$$W(\{\alpha_k\}, s) = \int \prod_k \exp - \left(\frac{2|\alpha_k - \beta|^2}{t-s} \right) W(\{\beta_k\}, t)\, \mathrm{d}^2 \left\{ \frac{2}{\pi(t-s)} \beta_k \right\} \tag{4.1}$$

This formula enables one to write immediately the weight functions describing thermal and fully coherent fields.

4.1. Thermal radiation

The weight function specifying the P representation ($t = 1$) of the density operator is given as

$$W(\{\alpha_k\}, 1) = \prod_k \frac{1}{\langle n_k \rangle} \exp\left(\frac{-|\alpha_k|^2}{\langle n_k \rangle} \right) \tag{4.2}$$

and from equation (4.1) we obtain

$$W(\{\alpha_k\}, s) = \prod_k \frac{1}{\langle n_k \rangle + \frac{1}{2}(1-s)} \exp\left(- \frac{|\alpha_k|^2}{\langle n_k \rangle + \frac{1}{2}(1-s)} \right) \tag{4.3}$$

which arises from (4.2) when the average number of photons in the kth mode $\langle n_k \rangle$ is replaced by $\langle n_k \rangle + \frac{1}{2}(1-s)$ (Perina 1971).

The characteristic functional is evaluated from equation (3.15):

$$\Phi(\{\zeta_k\}, s) = \prod_k \Phi_k(\zeta_k, s) \tag{4.4}$$

where

$$\Phi_k(\zeta_k, s) = \exp\{ -|\zeta_k|^2 [\langle n_k \rangle + \tfrac{1}{2}(1-s)] \}. \tag{4.5}$$

The s-ordered correlation function is

$$G_s^{(1,1)}(x_1, x_2) = \sum_k V^*(x_1, k) [\langle n_k \rangle + \tfrac{1}{2}(1-s)] V(x_2, k). \tag{4.6}$$

Thus, the characteristic functional becomes

$$\Phi[\zeta|s] = \exp\left(- \iint \zeta(x) G_s^{(1,1)}(x, x') \zeta^*(x')\, \mathrm{d}x\, \mathrm{d}x' \right) \tag{4.7}$$

and the probability functional is obtained by the Fourier inversion formula (3.20)

$$P[V|s] = \prod_k \frac{1}{\pi \lambda_k} \exp\left(- \iint G_s^{-1}(x, x') V(x) V^*(x')\, \mathrm{d}x\, \mathrm{d}x' \right) \tag{4.8}$$

where λ_k are the eigenvalues of the kernel $G_s(x, x')$ and $G_s^{-1}(x, x')$ is reciprocal kernel.

4.2. Fully coherent fields

In this case we have

$$W(\{\alpha_k\}, 1) = \prod_k \pi\delta^{(2)}(\alpha_k - \beta_k) \tag{4.9}$$

and from (4.1) we obtain

$$W(\{\alpha_k\}, s) = \prod_k \frac{2}{1-s} \exp\left(-\frac{2|\alpha_k - \beta_k|^2}{1-s}\right). \tag{4.10}$$

Formally, this is a function specifying the P representation ($s = 1$), corresponding to the superposition of coherent and thermal fields with the mean number of thermal photons in the kth mode equal to $1-s$. The characteristic functional is

$$\Phi(\{\zeta_k\}s) = \prod_k \Phi_k(\zeta_k, s) \tag{4.11}$$

where

$$\Phi(\zeta_k, s) = \exp(\zeta_k\beta_k^* - \zeta_k\beta_k)\exp\left(-\frac{1-s}{2}|\zeta_k|^2\right). \tag{4.12}$$

Equivalently, we can write

$$\Phi[\zeta|s] = \exp\left(\int \zeta(x)V^*(x, \{\beta_k\}\,\mathrm{d}x - \int \zeta(x)V(x\{\beta_k\})\,\mathrm{d}x\right)$$
$$\times \exp\left(-\int\int \zeta(x)\Gamma_s(x, x')\zeta^*(x')\,\mathrm{d}x\,\mathrm{d}x'\right) \tag{4.13}$$

where

$$\Gamma_s(x, x') = \sum_k V^*(x, k)\frac{1-s}{2}V(x'k). \tag{4.14}$$

5. Photon-number and photon-counting distributions

The developed formalism affords a straightforward method of dealing with the distributions of photons and photoelectrons. The probability for an arbitrary distribution of photons $\{n_k\}$ among the different modes is given by the expectation value of the projection operator

$$P(\{n_k\}) = \mathrm{Tr}[\rho|\{n_k\}\rangle\langle\{n_k\}|]. \tag{5.1}$$

Making use of (3.9) this becomes

$$p(\{n_k\}) = \int W(\{\alpha_k\}, s)\langle\{n_k\}|T(\{\alpha_k\}, -s)\{n_k\}\rangle\,\mathrm{d}^2\{\alpha_k/\pi\}. \tag{5.2}$$

The matrix elements of the operator $T(\alpha_k, -s)$ are explicitly given by (Cahill and Glauber 1969a)

$$\langle n_k|T(\alpha_k, -s)|n_k\rangle = \frac{2}{1+s}\left(\frac{s-1}{s+1}\right)^{n_k} \exp\left(-\frac{2|\alpha_k|^2}{1+s}\right)L_{n_k}\left(\frac{4|\alpha_k|^2}{1-s^2}\right). \tag{5.3}$$

Here and in the following the Laguerre polynomials are defined by the generating function

$$(1-z)^{-\alpha-1}\exp\left(\frac{xz}{z-1}\right) = \sum_{n=0}^{\infty} L_n^{\alpha}(x)z^n. \tag{5.4}$$

From (5.2) and (5.3) we finally obtain

$$p(\{n_k\}) = \int W(\{\alpha_k\}, s)\prod_k \left(\frac{s-1}{s+1}\right)^{n_k} \exp\left(-\frac{2|\alpha_k|^2}{1+s}\right)L_{n_k}\left(\frac{4|\alpha_k|^2}{1-s^2}\right)\prod_k \frac{2}{(1+s)\pi}\,\mathrm{d}^2\alpha_k. \tag{5.5}$$

The probability distribution $p(n)$ of the number of photons n within a given volume V of space at time t is obtained by summing (5.5) over all possible sets $\{n_k\}$

$$P_{\mathrm{phot}}(n) = \sum_{\{n_k\}} p(\{n_k\}) \tag{5.6}$$

with the subsidiary condition $\Sigma_k n_k = n$.

With the help of the addition theorem for the Laguerre polynomials (Gradshteyn and Ryzhik 1965)

$$\sum_{\{n_k\}} \prod_{k=1}^{M} L_{n_k}(x_k) = L_n^{M-1}(x_1 + \ldots + x_M) \tag{5.7}$$

we get

$$p_{\mathrm{phot}}(n) = \int W(\{\alpha_k\}, s)\left(\frac{s-1}{s+1}\right)^n \exp\left(-\frac{2U}{1+s}\right)L_n^{M-1}\left(\frac{4U}{1-s^2}\right)^2 \prod_{k=1}^{M} \frac{2}{(1+s)\pi}\,\mathrm{d}^2\alpha_k \tag{5.8}$$

where M is the number of modes and

$$U = \sum_{k=1}^{M} |\alpha_k|^2.$$

If we introduce the distribution function

$$W(U, s) = \int W(\{\alpha'_k\}, s)\delta\left(U - \sum_k |\alpha'_k|^2\right)\mathrm{d}^2\{\alpha_{k'}/\pi\}. \tag{5.9}$$

Equation (5.9) may be written in the form

$$p_{\mathrm{phot}}(n) = \left(\frac{2}{1+s}\right)^M \int_0^{\infty} W(U, s)\left(\frac{s-1}{s+1}\right)^N \exp\left(-\frac{2U}{1+s}\right)L_n^{M-1}\left(\frac{4U}{1-s^2}\right)\mathrm{d}U. \tag{5.10}$$

Equation (5.10), derived by Perina and Horak (1969, 1970), is termed the generalized photodetection equation.

The factorial moment generating function of $P_{\mathrm{phot}}(n)$, defined by

$$G_{\mathrm{phot}}(\lambda) = \sum_{n=0}^{\infty} (1-\lambda)^n P_{\mathrm{phot}}(n) \tag{5.11}$$

by virtue of (5.10) and (5.4) has the form

$$\begin{aligned} G_{\mathrm{phot}}(\lambda) &= [1-\tfrac{1}{2}(1-s)\lambda]^{-M}\left\langle \exp\left(\frac{\lambda U}{1-\frac{1}{2}(1-s)\lambda}\right)\right\rangle_s \\ &= [1-\tfrac{1}{2}(1-s)\lambda]^{-M}\int_0^{\infty} W(U, s)\exp\left(-\frac{\lambda U}{1-\frac{1}{2}(1-s)\lambda}\right)\mathrm{d}U. \end{aligned} \tag{5.12}$$

In the case of thermal light, equation (5.12) may be used as a starting point of the Mandel type approximation for the distribution of photons (Zardecki *et al* 1973).

The factorial moment generating function of the photocount distribution $p(n)$, defined by a formula analogous to (5.11) is given in terms of the normally ordered ($s = 1$) correlation functions (Glauber 1965).

$$G(\lambda) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \int \ldots \int G_1^{(n,m)}(x_1', \ldots, x_n', x_n'', \ldots, x_1'') \prod_{j=1}^{M} R(x_j', x_j'')\, dx_j'\, dx_j'' \tag{5.13}$$

where $R(x', x'')$ is the photodetector response function.

In order to express $G(\lambda)$ in terms of arbitrarily ordered correlation functions, we use the relation

$$G_1^{(n,n)}(x_1, \ldots, x_n; x_{n+1}, \ldots, x_{2n}) = \sum_{l=0}^{M} \binom{n}{l}\binom{n}{l}\left(\frac{s-1}{2}\right)^l \sum_P \prod_{j=1}^{l} K(x_{p(j)} x_j) G_s^{n-l,n-l}(x_{l+1}, \ldots, x_n; x_{n+l-1}, \ldots, x_{2n}) \tag{5.14}$$

which follows from (3.26).

When the correlation functions on the right-hand side of (5.14) are written in the form of (3.22), we obtain on combining this with (5.13)

$$G(\lambda) = P[V|s] \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \sum_{l=1}^{n} l! \binom{n}{l}\binom{n}{l}\left(\frac{s-1}{2}\right)^l \kappa^l \Omega^{n-l}\, d^2 V \tag{5.15}$$

where

$$\begin{aligned} \kappa &= \iint K(x', x'')R(x', x'')\, dx'\, dx'' \\ \Omega &= \iint V^*(x')R(x', x'')V(x^*)\, dx'\, dx''. \end{aligned} \tag{5.16}$$

We now use the identity (Cahill and Glauber 1969a)

$$\begin{aligned} \sum_{r=0}^{(n,m)_{\min}} l! \binom{n}{l}\binom{m}{l} v^l \lambda^{n-l} \mu^{m-l} &= n! v^n \mu^{m-n} L_n^{(m-n)}\left(-\frac{\lambda\mu}{v}\right) \\ &= m! v^m \lambda^{n-m} L_m^{(n-m)}\left(-\frac{\lambda\mu}{v}\right) \end{aligned} \tag{5.17}$$

to transform (5.11) to the form

$$G(\lambda) = \int P[V|s] \sum_{n=0}^{\infty} \left(\lambda\kappa \frac{1-s}{2}\right)^n L_n\left(\frac{2\Omega}{(1-s)\kappa}\right) d^2 V. \tag{5.18}$$

Making use of (5.4) we find

$$G(\lambda) = \int P[V|s] \exp\left(-\frac{\lambda\Omega}{1-\frac{1}{2}\lambda\kappa(1-s)}\right) d^2\left(\frac{V}{1-\frac{1}{2}\lambda\kappa(1-s)}\right). \tag{5.19}$$

For a broad-band detector and plane quasi-monochromatic waves normally incident on a photocathode, Ω becomes proportional to the integrated light intensity (Perina 1971)

$$\Omega = \alpha W = \alpha \int_0^T I(t)\,\mathrm{d}t \tag{5.20}$$

where α denotes the detector photosensitivity.

Equation (5.19), with Ω given by (5.20), becomes identical with equation (5.8) in the case of a finite number of M modes, if we put $\alpha = \kappa = 1$. In fact, the functional integral in (5.19) goes over into an M-dimensional integral and we have

$$G(\lambda) = [1-\tfrac{1}{2}\lambda(1-s)]^M \left\langle \exp\left(-\frac{\lambda\alpha W}{1-\frac{1}{2}(1-s)\lambda}\right)\right\rangle_s. \tag{5.21}$$

If we put $s = 1$, this yields the well known formula for the normally ordered generating function

$$G_{\mathrm{norm}}(\lambda) = \langle \exp(-\lambda\alpha W)\rangle_1$$
$$= \int P[V|1] \exp(-\lambda\alpha W)\,\mathrm{d}^2 V. \tag{5.22}$$

It is seen that apart from a constant factor, equation (5.19) coincides formally with (5.22) if we replace α in (5.22) by $\alpha/[1-\frac{1}{2}\lambda\kappa(1-s)]$ and make a corresponding change in the integration element. This formal similarity permits us to write immediately a formula expressing $G(\lambda)$ through the characteristic functional of the field (Zardecki 1971):

$$G(\lambda) = \int \Phi\left[\left(\frac{\lambda\alpha}{1-\frac{1}{2}\lambda\kappa(1-s)}\right)^{1/2}\zeta|s\right] \exp[-(\zeta^*,\zeta)]\,\mathrm{d}^2\left(\frac{\zeta}{\pi[1-\frac{1}{2}\lambda\kappa(1-s)]}\right)$$
$$= \int \Phi[(\lambda\alpha)^{1/2}\zeta|s] \exp\{[1-\tfrac{1}{2}\lambda\kappa(1-s)](\zeta^*,\zeta)\}\,\mathrm{d}^2(\zeta/\pi). \tag{5.23}$$

For Gaussian fields the characteristic functional is given by (4.7) and the functional integration in (5.23) can be effectively performed. The results corresponding to various physical situations derived earlier for $s = 1$ (Zardecki 1971) are readily extended for arbitrary ordering.

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